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A New Coordinate Condition, and Elementary Flatness, for Axially Symmetric Interior Solutions in General Relativity

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Abstract

Two general results for stationary axially symmetric interior solutions of the Einstein or Einstein–Maxwell equations in cylindrical coordinates are derived.

Firstly, a coordinate condition for interior solutions is proposed, corresponding to the Weyl coordinate condition used in the exterior.

Secondly, it is shown that elementary flatness in the interior is always ensured by realistic boundary conditions and matter tensors, given elementary flatness in the exterior metric.

A physical discussion of the results is given, particularly in reference to solutions which have singular struts in them.

1. *Geometrical Preliminaries*

It is assumed for simplicity that space-time is occupied by a single, compact, isolated body, surrounded by vacuum or electromagnetic fields, and that the entire system is stationary and axially-symmetric.

Thus, space-time is divided into two regions, I and E, the regions interior and exterior to the body, respectively. Each region has a patch of coordinates $x^\mu = (z, r, \phi, t)$ with $0 \leqslant \phi \leqslant 2\pi, -\infty \leqslant t \leqslant \infty, \mu = 1, 2, 3, 4$. Then, all functions are independent of ϕ and t, and in each hypersurface $t = cst$, there is rotational symmetry about the axis $r = 0$. I and E are separated by a time-like 3-surface B, the history of the boundary of the body.

B is defined parametrically in I and E separately, using parameters $X^A = (\theta, \phi, t)$, $A = 0, 3, 4$, which act as coordinates on B:

$$\underset{I}{B} \stackrel{d}{=} \{x^\mu = \underset{I}{f^\mu}(X^A)\colon z = \underset{I}{\mathbf{z}}(\theta),\ r = \underset{I}{\mathbf{r}}(\theta),\ \phi = \phi,\ t = t\}$$

and similarly in E. θ takes the values $0 \leqslant \theta \leqslant \pi$, and $\underset{I}{\mathbf{z}}, \underset{I}{\mathbf{r}}, \underset{E}{\mathbf{z}}, \underset{E}{\mathbf{r}}$, are C^2 functions of θ. In order that B be a closed two surface in each hypersurface $t = cst$, $r(\theta) \to 0$ as $\theta \to 0, \pi$ is necessary. Thus

$$\mathbf{r}(\theta) = \sin\theta\rho(\theta), \quad \rho(\theta) \in C^2$$

$\rho(\theta) > 0$ is necessary, for all θ. For $0 < \theta < \pi$, it is necessary to ensure that the body has finite thickness, whilst at the poles $\theta = 0, \pi$ it is necessary in order that the tangent in the z, r plane be defined there (see below).

Then, I is the set of points x^μ such that the pairs (z, r) lie within and on the closed curve formed by the arc $z = \underset{I}{\mathbf{z}}(\theta)$, $r = \underset{I}{\mathbf{r}}(\theta)$, and the axis $\underset{I}{\mathbf{z}}\min \leqslant z \leqslant \underset{I}{\mathbf{z}}\max$, $r = 0$, in the Euclidean half-plane $-\infty \leqslant z \leqslant \infty$, $r \geqslant 0$, whilst E is the set of points x^μ such that the pairs (z, r) lie on and outside the arc $z = \underset{E}{\mathbf{z}}(\theta)$, $r = \underset{E}{\mathbf{r}}(\theta)$, in the same half-plane.

The un-normalised tangents $\tau_A{}^\mu$ to B, A acting as a counting index, are defined by

$$\tau_A{}^\mu \overset{d}{=} \frac{\partial f^\mu(X^A)}{\partial X^A}$$

in I and E separately. Thus

$$\tau_0{}^\mu = (\mathbf{z}_{,\theta}, \mathbf{r}_{,\theta}, 0, 0)$$
$$\tau_3{}^\mu = \delta_3{}^\mu, \quad \tau_4{}^\mu = \delta_4{}^\mu$$

in I and E separately. (The comma is used to denote total or partial differentiation as appropriate.) In order that B be smooth at the poles

$$\tau_0{}^1 \to 0 \quad \text{as} \quad \theta \to 0, \pi$$

Then, as $r \to 0$ on B in I and E separately,

$$\frac{\underset{I}{\tau_0{}^2}(\theta)}{\underset{E}{\tau_0{}^2}(\theta)} \to \frac{\underset{I}{\mathbf{r}}(\theta)}{\underset{E}{\mathbf{r}}(\theta)} \to \frac{\underset{I}{\rho}(0)}{\underset{E}{\rho}(0)}$$

as $\theta \to 0$ and similarly as $\theta \to \pi$.

In I and E separately, it is assumed that Einstein's equation

$$G_\mu{}^\nu = -\kappa T_\mu{}^\nu$$

may be solved for a metric in the form

$$ds^2 = g_{11}(dz^2 + dr^2) + g_{33}\, d\phi^2 + 2g_{34}\, d\phi\, dt + g_{44}\, dt^2 \tag{1.1}$$

where $g_{\mu\nu}(z, r) \in C^3$ and $T_\mu{}^\nu(z, r) \in C^1$ in the appropriate region. It is also assumed that the boundary conditions (Darmois, 1927)

$$[h_{AB}]_B = 0 \tag{1.2}$$

may be satisfied by $g_{\mu\nu}$. Here, h_{AB} is the induced first fundamental form on B:

$$\underset{I}{h_{AB}}(X^c) \overset{d}{=} \underset{I}{g_{\mu\nu}}(x^\mu = \underset{I}{f^\mu}(X^c))\, \underset{I}{\tau_A{}^\mu}\, \underset{I}{\tau_B{}^\nu}$$

and similarly from E. Also, for $\underset{E}{F}(x^\mu)$ and $\underset{I}{F}(x^\mu)$ arbitrary, $[F]_B$ is the discontinuity of F across B:

$$[F]_B \stackrel{d}{=} \underset{E}{F}(x^\mu = \underset{E}{f^\mu}) - \underset{I}{F}(x^\mu = \underset{I}{f^\mu})$$

In E, it is assumed that elementary flatness is satisfied (Synge, 1960, p. 313):

$$\underset{E}{\gamma_{33}} \stackrel{d}{=} \frac{1}{r^2} \underset{E}{g_{33}} : \frac{\underset{E}{\gamma_{33}}}{\underset{E}{g_{11}}} \to 1 \quad \text{as} \quad r \to 0$$

and that Weyl's coordinate condition is satisfied (Synge, 1960, p. 310; Lewis, 1932):

$$\underset{E}{\Delta} \stackrel{d}{=} -\underset{E}{g_{33}} \underset{E}{g_{44}} + \underset{E}{g^2_{34}} = r^2$$

Finally, it is assumed that g_{34}, in both I and E, is such that

$$\gamma_{34} \stackrel{d}{=} \frac{1}{r^2} g_{34}$$

is C^3 in the appropriate region.

2. *The Interior Coordinate Condition*

One of the field equations may be written as

$$G_1{}^1 + G_2{}^2 = -\kappa(T_1{}^1 + T_2{}^2) \tag{2.1a}$$

where

$$-\Delta^{1/2} g_{11}(G_1{}^1 + G_2{}^2) \stackrel{d}{=} D^2(\Delta^{1/2}) \tag{2.1b}$$

and

$$D^2 \stackrel{d}{=} \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2}$$

Since

$$\Delta^{1/2} = r\sqrt{(-\gamma_{33} g_{44} + r^2 \gamma_{34}^2)}$$

(2.1) imply that

$$2[\sqrt{(-\gamma_{33} g_{44})}]_{,r} \to 0 \quad \text{as} \quad r \to 0$$

Thus

$$\gamma_{33} g_{44} \to F(z)$$

where F is arbitrary*, as $r \to 0$, in both I and E. To choose a form for $\underset{I}{F}$, consider E, where Weyl's condition gives $\underset{E}{F} = -1$. One is free to choose $\underset{I}{F} = -1$.

* The reason that $F(z)$ in Section 2 is arbitrary is that (2.1) is the only field equation for g_{33}, in the sense that all the other field equations may be solved (in principle) for the other elements of $g_{\mu\nu}$ with g_{33} undetermined except for (2.1).

Thus the proposed coordinate system for use in I is: a metric of the form (1.1) and $x^\mu = (z, r, \phi, t)$ as explained in Section 1, with the coordinate condition

$$\underset{I}{\gamma_{33}}\,\underset{I}{g_{44}} \to -1 \quad \text{as} \quad r \to 0 \tag{2.2a}$$

Elementary flatness in I allows the alternative form

$$\underset{I}{g_{11}}\,\underset{I}{g_{44}} \to -1 \quad \text{as} \quad r \to 0 \tag{2.2b}$$

3. *Elementary Flatness in I*

One of the field equations may be written as

$$G_1{}^2 = -\kappa T_1{}^2 \tag{3.1a}$$

where

$$4\Delta^2 g_{11}^2\, G_1{}^2 \overset{d}{=} 2\Delta g_{11}\, \Delta_{,zr} - \Delta(g_{11,r}\Delta_{,z} + g_{11,z}\Delta_{,r}) - g_{11}\,\Delta_{,z}\Delta_{,r}$$
$$+ \Delta g_{11}(g_{33,z}g_{44,r} + g_{33,r}g_{44,z} - 2g_{34,z}g_{34,r}) \tag{3.1b}$$

Write Δ as

$$\Delta = r^2(-\gamma_{33}\, g_{44} + r^2\gamma_{34}^2)$$

Evaluating all the terms in the field equation (3.1b), one finds that the LHS $\sim O(r^4)$, whilst some terms in the RHS $\sim O(r^3)$. Thus, as $r \to 0$, these terms tend to zero, and the resulting equation is

$$g_{11}\,\gamma_{33,z} - g_{11,z}\,\gamma_{33} \to 0 \quad \text{as} \quad r \to 0$$

Thus

$$\frac{g_{11}}{\gamma_{33}} \to cst \quad \text{as} \quad r \to 0 \tag{3.2}$$

Now, (1.2) yields, as two of the metric boundary conditions,

$$\left.\begin{aligned} [g_{33}]_B \equiv [r^2\gamma_{33}]_B &= 0 \\ [g_{11}\{(\tau_0{}^1)^2 + (\tau_0{}^2)^2\}]_B &= 0 \end{aligned}\right\} \tag{3.3}$$

Applying the results of Section 1 to the conditions (3.3) yields

$$\left.\begin{aligned} (\underset{I}{\gamma_{33}})_{\underset{I}{B}} &\to \left(\frac{\underset{I}{\rho(0)}}{\underset{E}{\rho(0)}}\right)^2 (\underset{E}{\gamma_{33}})_{\underset{E}{B}} \\ (\underset{I}{g_{11}})_{\underset{I}{B}} &\to \left(\frac{\underset{I}{\rho(0)}}{\underset{E}{\rho(0)}}\right)^2 (\underset{E}{g_{11}})_{\underset{E}{B}} \end{aligned}\right\} \tag{3.4}$$

as $r \to 0$ on B, $\theta \to 0$, where $(\underset{E}{F})_{\underset{E}{B}}$ means $\underset{E}{F}(x^\mu = \underset{E}{f^\mu}(X^c))$ and similarly for I, and similarly as $r \to 0$ on B, $\theta \to \pi$.

Now elementary flatness in E ensures that

$$\frac{\underset{E}{\gamma_{33}}}{\underset{E}{g_{11}}} \to 1 \quad \text{as } r \to 0$$

throughout E, so that (3.4) ensures that

$$\left(\frac{\underset{I}{\gamma_{33}}}{\underset{I}{g_{11}}}\right)_{\underset{I}{B}} \to 1 \quad \text{as } r \to 0 \text{ on } B$$

and (3.2) ensures that

$$\frac{\underset{I}{\gamma_{33}}}{\underset{I}{g_{11}}} \to 1 \quad \text{as } r \to 0 \text{ throughout } I$$

Thus elementary flatness in E, satisfaction of the boundary conditions (3.3), and the field equation (3.1) in I, ensure elementary flatness in I. (This is independent of the choice of $\underset{I}{F}$ in Section 2.)

4. *Discussion of the Results*

The coordinate condition of Section 2 is derived from the same field equation as is Weyl's condition for E, and so (2.2) may be regarded as generalising Weyl's condition to the case that $(G_1{}^1 + G_2{}^2)$ does not vanish. It may therefore be used in finding interior solutions for the numerous axially symmetric exterior solutions which are known, and possibly also for developing cosmological-type solutions for matter fields which fill the whole of space-time.

In matching interior and exterior solutions, the boundary conditions (1.2) would be supplemented by (Darmois, 1927; Israel, 1966):

$$[k_{AB}]_B = 0 \tag{4.1}$$

where k_{AB} is the induced second fundamental form of B:

$$\underset{I}{k_{AB}} \stackrel{d}{=} -\underset{I}{n_{\mu;\nu}}\, \underset{I}{\tau_A{}^{\mu}}\, \underset{I}{\tau_B{}^{\nu}}$$

where $\underset{I}{n_\mu}$ is the unit normal to $\underset{I}{B}$ and similarly from E. With metrics of the form (1.1) in both E and I, which satisfy (1.2) and (4.1) across B, a sufficient condition for admissible coordinates ($g_{\mu\nu} \in C'$ across B) is that $[\tau_A{}^{\mu}] = 0$. In the light of zum Haagen (1969), one can conjecture the existence of solutions in the form (1.1) in I, for ranges of coordinates as described in Section 1, because the boundary conditions (1.2) and (4.1) are much less restrictive than those of zum Haagen, namely that $g_{\mu\nu} \in C^2$ across B.

The results of Section 3 are more interesting, physically, as the condition of elementary flatness on $r = 0$ corresponds to the absence of struts along the z-axis. See, for example, Synge (1960, p. 313), Bonnor (1969), Sackfield (1971) and Bonnor & Swaminarayan (1964).

The result of Section 3 means that a strut of tension or compression cannot exist inside an isolated body without a strut of some type outside the body, extending to infinity, given that the boundary conditions (1.2) are satisfied. A strut inside a body shows itself as singularities in the stresses as well as in the metric on $r = 0$ in I.

However, if (1.2) are relaxed, then a strut could be allowed inside the body, by having equal and opposite discontinuities in h_{AB} at the poles, and no struts outside would be needed. If the discontinuities at the poles were to be of different magnitudes, then struts outside would be necessary, though one could be dispensed with by suitable choice of the discontinuity at that pole.

In the case of two or more bodies with struts between them, or for accelerated bodies, where the struts appear to exert forces on the bodies, there appear to be two ways of accommodating the forces. Either, (1.2) can be relaxed, at (and perhaps near) appropriate poles, or (1.2) can be maintained whilst allowing a singular stress system along $r = 0$ in the I of appropriate bodies.

The meaning of a rotating spike—a singularity on $r = 0$ of angular momentum rather than stress (Bonnor, 1969; Sackfield, 1971)—could be that $[h_{34}] \neq 0$ at a pole.

The meaning of $[h_{AB}] \neq 0$ is hinted at in Bonnor & Sackfield (1968, Section 4). There it is shown that $[h_{44}] \equiv [g_{44}] \neq 0$ corresponds to a dipole layer of mass, so that $[h_{AB}] \neq 0$ seems to correspond to the existence of double layers of mass and stress, and the relaxing of (1.2) at and near a pole would correspond to allowing such layers to spread the load of the strut into the interior of the body. This boundary condition has not been much discussed.

It is interesting that the boundary condition (4.1) does not enter into the discussion. If (4.1) is relaxed, then as shown in Israel (1966), single layers of mass and stress appear on B, but these seem to play no part in the action of struts on bodies. However, as the field equations have not been solved for such systems, one cannot be certain that $[h_{AB}] \neq 0$, $[k_{AB}] = 0$, are compatible boundary conditions.

For realistic solutions, the struts would be replaced by thick columns, and all such connected bodies would form a single non-singular I of the type discussed in Sections 1 to 3.

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